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The Solutions to a Smooth PDE Can Be Dense in $C[I]$*

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The solutions to a partial differential equation that arises in a physical context are expected to be restricted in nature since they are intended to represent some physically significant behavior. In particular, they will approximate an arbitrary continuous function f on a compact set K only if K is "thin"—e.g., nowhere dense in $\mathbb{R}^n$. The purpose of this note is to show that this does not hold for all partial differential equations, even if they are required to be very smooth. Specifically, it will be shown that for any $n \geqslant 2$, there exist partial differential equations of polynomial type on $\mathbb{R}^n$ whose C^∞ solutions are uniformly dense in the space $C[I]$ of all continuous real functions on the n-cube I.

This result also may possibly be the first concrete application of the Kolmogorov–Arnold solution of the Hilbert 13th Problem (see [3]).

We first set the notation. I is the set of all $x = (x_1, x_2,..., x_n) \in \mathbb{R}^n$, with $0 \leqslant x_i \leqslant 1$. The vector exponent $\alpha = (\alpha_1, \alpha_2,..., \alpha_n)$ has integer components $\alpha_i \geqslant 0$, and $|\alpha|$ denotes $\sum_1^n \alpha_i$. The general multinomial is denoted by

$$x^\alpha = \prod_1^n (x_i)^{\alpha_i}$$

and is said to have (total) degree $|\alpha|$. We use a similar notation for partial differentiation, so that

$$\left(\frac{\partial}{\partial x}\right)^\alpha F = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n} F$$

denotes one of the derivatives of F of order $|\alpha|$.

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MAIN THEOREM. *If $n \geqslant 2$ there are integers $k = k(n)$ and $\Delta = \Delta(n)$, and a partial differential equation on $\mathbb{R}^n$ of order not greater than k, whose C^∞ solutions $F(x)$ are uniformly dense in $C[I]$, the space of continuous real functions on the n-cube. The form of the differential equation is*

$$Q(w_1, w_2, \dots, w_N) = 0, \tag{1}$$

where $w_1, w_2, \dots, w_N$ is a listing of all the partial derivatives of F of order $|\alpha| \leqslant k$, and Q is a non-trivial real polynomial of total degree at most Δ.

Explicit values of k and Δ can be found for any $n \geqslant 2$. However, since these numbers are large when n is small, it is not likely that anyone will wish to exhibit Q itself. Thus, the interest in this result lies in the fact that such "universal" differential equations exist, having C^∞ solutions with the uniform approximation property, and that the PDE can be taken to be of *polynomial type*.

The construction of Q will depend upon a combination of familiar facts, of which the central one is the Kolmogorov solution of Hilbert's 13th problem (see [3]). The solution asserts that every real continuous function F on I can be represented there in the form

$$F = \sum_{1}^{2n+1} f_j \circ g_j, \tag{2}$$

where the f_j are continuous functions on $\mathbb{R}$ to $\mathbb{R}$, and the g_j are sums of such functions:

$$g_j(x) = {}_j\phi_1(x_1) + {}_j\phi_2(x_2) + \cdots + {}_j\phi_n(x_n). \tag{3}$$

Here, ${}_j\phi_i$ is some continuous function on R to R.

The known proofs of this surprising result, which reversed Hilbert's conjecture, are non-constructive and depend upon the Baire category theorem. However, it is known that one cannot in general hope to have all the component functions f_j and ${}_j\phi_i$ of class C^1, even if F itself is C^∞.

We introduce the term *smooth Kolmogorov function* for a function F defined on I that can be represented there in the formats (2) and (3), with each of the component functions f_j and ${}_j\phi_i$ is of class C^∞.

LEMMA 1. *The class of smooth Kolmogorov functions is a uniformly dense subset of $C[I]$.*

We prove the main theorem by showing that we can construct a PDE of the form (1) whose solutions include all the smooth Kolmogorov functions. We need two elementary results dealing with polynomials. The first is simple counting.

LEMMA 2. *The polynomials in r indeterminates, with total degree at most d, form a vector space $V(r, d)$ of dimension*

$$\binom{r+d}{r}.$$

The second is a specific form of a standard elimination theorem for polynomials.

LEMMA 3. (i) *If $N > m$ and if $P_1(u),..., P_N(u)$ is a collection of N polynomials in the indeterminates $u = (u_1, u_2,..., u_m)$, then there is a non-trivial polynomial Q in N indeterminates such that*, w.r.t. u,

$$Q(P_1, P_2,..., P_N) = 0. \tag{4}$$

(ii) *If* $\deg(P_i) \leqslant \sigma$ *for each i, then*

$$\deg(Q) \leqslant N\sigma^{m/(N-m)}. \tag{5}$$

Proof. Put $w_i = P_i(u)$, for $i = 1, 2,..., N$. The idea is to treat these as a collection of non-linear algebraic equations, from which we eliminate the $u_1, u_2,..., u_m$ to obtain (4). Choose an integer Δ, as yet unspecified, and form all the multinomials w^α for $|\alpha| \leqslant \Delta$. By Lemma 2, this produces a collection of size $s(\Delta)$, where

$$s(\Delta) = \binom{\Delta + N}{N}. \tag{6}$$

Note that

$$w^\alpha = \prod_{i=1}^{N} P_i(u_1, u_2,..., u_m)^{\alpha_i}, \tag{7}$$

which is a polynomial in the indeterminates u_i of total degree at most $\sum \sigma\alpha_i = \sigma|\alpha| \leqslant \sigma\Delta$, and belonging to the vector space $V(m, \sigma\Delta)$. By Lemma 2, the dimension of this space is

$$t(\Delta) = \binom{\sigma\Delta + m}{m}. \tag{8}$$

If Δ is chosen so that $s(\Delta) > t(\Delta)$, then the polynomials w^α will be linearly dependent, and there will exist scalars c_α, not all zero, such that

$$\begin{aligned} 0 = \sum_\alpha c_\alpha w^\alpha &= \sum_\alpha c_\alpha \prod_1^N w_i^{\alpha_i} \\ &= Q(w_1, w_2,..., w_N). \end{aligned}$$

We have thus produced the required polynomial, whose degree is at most Δ. To see that we can choose Δ so that $s(\Delta) > t(\Delta)$, we observe that by (6), $s(\Delta) = O(\Delta^N)$ while (8) shows that $t(\Delta) = O(\Delta^m)$. Since $N > m$, we have $s(\Delta) > t(\Delta)$ when Δ is sufficiently large. To obtain the explicit estimate (5), we note

$$s(\Delta) > (\Delta/N)^{N-m} \sigma^{-m} t(\Delta).$$

We begin the proof of the Main Theorem by returning to (2) and differentiating it repeatedly, with respect to the underlying variables x_i, up to partial derivatives of total order at most k, thereby obtaining a large number of equations of the form

$$\left(\frac{\partial}{\partial x}\right)^{\alpha} F = F_{\alpha} = P_{\alpha}(u_1, u_2, \dots, u_m), \tag{9}$$

where the u_i are the functions that arise from differentiating the component functions f_j and ${}_j\phi_i$, and the P_α are polynomials.

This may be clarified by considering a related case (see [1]). Suppose that

$$\begin{aligned} G(x_1, x_2) &= f(\phi(x_1) + \psi(x_2)) \\ &= (f \circ g)(x), \end{aligned} \tag{10}$$

where f, ϕ, and ψ are in C^∞. Differentiating, we obtain

$$\begin{aligned} G_{1,0} &= u_1 u_2, \\ G_{0,1} &= u_1 u_3, \\ G_{2,0} &= u_1 u_5 + u_4 (u_2)^2, \\ G_{1,1} &= u_4 u_2 u_3 \end{aligned}$$

etc., where

$$\begin{aligned} u_1 &= f' \circ g, \qquad & u_2 &= \phi', \qquad & u_3 &= \psi', \\ u_4 &= f'' \circ g, \qquad & u_5 &= \phi'', \qquad & u_6 &= \psi'', \end{aligned}$$

etc. In this, if one eliminates the u_i, the result is a polynomial relation among the various G_α, namely:

$$(G_{10})^2 G_{01} G_{12} - G_{10} (G_{01})^2 G_{21} - (G_{10})^2 G_{11} G_{02} + (G_{01})^2 G_{11} G_{20} = 0.$$

This is a partial differential equation of order 3, degree 4, of polynomial type satisfied by all the functions G of two variables that can be represented smoothly in the format (10). Note that the equation itself does not depend upon the component functions f, ϕ, or ψ, but only upon the format.

We apply the same procedure to (2). Differentiating this up to order k, we produce N equations of the form

$$F_\alpha = P_\alpha(u_1 u_2,\ldots, u_m), \qquad 1 \leqslant |\alpha| \leqslant k,$$

where the u_i are the functions $D^p{}_j\phi_i$ and $(D^p f_j) \circ g_j$ for $p = 1, 2,\ldots, k$; $j = 1, 2,\ldots, 2n+1$; $i = 1, 2,\ldots, n$; and $D = d/dt$. We see that $m = k(2n+1)(n+1)$. Since we do not include the value $\alpha = 0$, the number of equations obtained is one less than the dimension of $V(n, k)$. Thus,

$$N = \binom{k+n}{n} - 1.$$

Each of the polynomials P_α has degree at most $k+1$, and has integral coefficients. Since $m = O(k)$ and $N = O(k^n)$ we see that $N > m$ when $n \geqslant 2$ and k is sufficiently large. Thus, by Lemma 3, there is a polynomial Q having rational coefficients whose degree is not larger than the integer Δ given in (5), and such that

$$Q\left(\ldots, \left(\frac{\partial}{\partial x}\right)^\alpha F, \ldots\right) = 0, \tag{11}$$

holding for any function F that is smoothly represented in the format (2), (3); moreover, Q does not depend upon the choices of the component functions f_j and ${}_j\phi_i$. Thus, (11) is a partial differential equation of polynomial type that is satisfied by all the smooth Kolmogorov functions of n variables defined on the cell I. Since these are dense in $C[I]$, the theorem is proved.

For the record, we note that if $n = 2$, we can choose $k = 28$, $N = 434$, and $m = 420$; if $n = 10$, take $k = 4$, $N = 1000$, and $m = 924$; if $n = 32$, take $k = 3$, $N = 6544$, and $m = 6435$. In all these cases, the upper bound estimate for Δ is very large.

There is no reason not to hope for "universal" PDE of much lower order, degree, and dimension. The present approach does not yield one of polynomial type for order $k = 2$.

The restriction to polynomial type may be critical for such examples. In subsequent discussions, Professors Crandall and Turner have shown that one can show the existence of differential equations of the form $Q(u', u'') = 0$, whose solution sets are uniformly dense on intervals, and with Q a non-polynomial function of class C^∞ [2]. These can arise by careful selection in advance of functions u to belong to the solution set; the function Q is then chosen so that it vanishes on each of a countable family of curves γ_u defined by

$$\gamma_u(t) = (u'(t), u''(t))$$

for $0 \leqslant t \leqslant 1$. Each of these functions u is then automatically a solution of $Q(u', u'') = 0$. In particular this device can be used on any function Q with compact support, and the curves γ_u chosen to lie outside the support of Q, but such that the functions u are dense in $C[0, 1]$. Such a function Q, of course, will not be a polynomial.

Note added in proof. Since the results in this paper were announced at the Northwestern Approximation Conference in October 1979, L. R. Rubel has shown that the Crandall–Turner results for ODE can be improved by choosing Q to be a polynomial, and R. J. Duffin has exhibited a 4th order polynomial ODE whose solutions include a class of splines sufficient to approximate any continuous function on $(-\infty, \infty)$.

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